Int. J. Heat Mass Transfer. Vol. 22, pp. 1331–1337
Pergamon Press Ltd. 1979. Printed in Great Britain

ON AN IMPLICIT SCHEME FOR THE ISOTHERM MIGRATION METHOD ALONG ORTHOGONAL FLOW LINES IN TWO DIMENSIONS

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(*Received* 20 *September* 1978 *and in revised form* 27 *February* 1979)

Abstract—In a previous paper the authors developed an explicit scheme for the numerical solution of the equations of isotherm migration along orthogonal flow lines in two space dimensions. This paper presents an implicit scheme. It is used to solve a two-dimensional one-phase Stefan problem and results are compared with those obtained by other methods.

NOMENCLATURE

L, non-dimensional latent heat;
Δ, $n_{j,m}$, $|r_{j,m}|$;
$n_{j,m}$, increment in $n_{j,m}$ during time step Δt;
r, radical coordinate;
$r_{j,m}$, radius of curvature of isotherm $j\,\delta u$ at point m (see Section 3);
s_m, length of chord m;
$S(x, y, t) = 0$, freezing front;
t, non-dimensional time;
Δt, time step;
u, non-dimensional temperature;
δu, temperature step;
x, y, cartesian coordinates.

Greek symbols

γ_m, half-angle subtended at centre of curvature by chord m (see Section 3);
θ, angular coordinate;
ψ_m, angle between perpendicular bisector of chord m and x axis (see Section 3).

Subscripts

j, pertaining to isotherm with temperature $j\delta u$;
m, pertaining to point (or chord) m.

1. INTRODUCTION

IN THIS paper we consider a reformulation of problems of two-dimensional heat conduction in an isotropic medium. In such a material the heat flow is always normal to an isotherm. Thus there exists a natural set of othogonal curvilinear coordinates for the problem, one family of coordinate lines being the isotherms, and the other the lines of heat flow. Following the ideas of Dix and Cizek [1] Crank and Phale [2] we seek to exploit the orthogonality of the isotherms and flow lines, in order to write the partial differential equation of heat conduction in a locally one-dimensional, but non-linear, form, and hence track the motion of such isotherms. This is analogous to the Lagrangian formulation of fluid flow, in which the motion of particular particles of fluid is calculated, rather than the velocity distribution at fixed points in space. In the reformulated equation the independent variables are the temperature and time. This isotherm migration method (IMM) is suitable for phase change problems, in which the phase change interface is an isotherm, and is especially useful if the position of the interface, rather than the temperature distribution, is of particular interest.

In a recent paper [3] the IMM along orthogonal flow lines was introduced, and applied to a simple two-dimensional problem, using an explicit finite-difference scheme to solve the derived equations. Here we describe an implicit numerical technique for a linearised form of the partial differential equation, which permits longer time-steps to be used, with a corresponding increase in computational efficiency. The method is used to solve a more exacting problem than that in [3], and the results are compared with those obtained using the scheme described in [3], and those of other authors.

2. IMM REFORMULATION OF TWO-DIMENSIONAL HEAT CONDUCTION

In cylindrical polar coordinates (r, θ) the partial differential equation of heat conduction may be written, in non-dimensional variables, as

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2}, \tag{1}$$

where u denotes the temperature, and the thermal properties are assumed constant. Following [1], the change of dependent variable from $u(r, \theta, t)$ to $r(u, \theta, t)$ yields the differential equation

$$\frac{\partial r}{\partial t} = \frac{\partial^2 r}{\partial u^2} \Big/ \left(\frac{\partial r}{\partial u}\right)^2 - \frac{1}{r} - \frac{1}{r^2}\frac{\partial r}{\partial u}\frac{\partial^2 u}{\partial \theta^2}. \tag{2}$$

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In two-dimensional heat conduction in an isotropic material, the isotherm at any point moves normal to itself, along the heat flow lines. For a short interval of time, we may regard the centre of curvature of each segment of an isotherm as remaining fixed, and identify the local radius of curvature with r in equation (2), and the centre of curvature with the origin of a purely *local* set of plane polar coordinates. Thus equation (2) may be used to calculate the local velocity $(\partial r/\partial t)$ of the isotherm normal to itself, along its radius of curvature.

In a general heat flow problem the set of isotherms will rotate, and the centre of curvature will move with time. Thus the local centre and radius of curvature must be recalculated at each time interval by a geometrical procedure. However, provided the time steps are sufficiently small for the movement of the centre of curvature during any one step to be negligible, (2) reduces to

$$\frac{\partial r}{\partial t} = \left(\frac{\partial^2 r}{\partial u^2}\right) \Big/ \left(\frac{\partial r}{\partial u}\right)^2 - \frac{1}{r}, \tag{3}$$

once r is identified as the local radius of curvature. For a fuller discussion of this derivation see [3].

The procedure to obtain a numerical solution thus involves two distinct stages: firstly, a geometrical calculation to determine the local centres and radii of curvature at points along each of a set of isotherms; and secondly, the solution of (3), and the calculation of the resulting motion of each isotherm along the normal to itself. We describe the geometrical procedure in the next section, and then discuss the solution of the differential equation in Section 4. In Section 5, the application of the method to moving boundary problems is outlined, before a numerical example is solved in Section 6.

3. THE CALCULATION OF THE LOCAL COORDINATES

Consider the segment of the isotherm ABC (shown in Fig. 1) which we regard as a circular arc, concave downwards, and to which we assign positive curvature. The tangents at the midpoints P, Q of each of the arcs AB, BC are parallel to the corresponding chords AB, BC. Thus the change in direction of the tangent along the arc QBP is given by $(\psi_m - \psi_{m+1})$, where $\psi_m(\psi_{m+1})$ is the angle the perpendicular bisector of the chord $AB(BC)$ makes with the x axis. Then the radius of curvature at the point B with coordinates (x_m, y_m) may be written as

$$r_m = \left(\frac{\partial s}{\partial \psi}\right)_m = \frac{s_{m+1} + s_m}{2(\psi_m - \psi_{m+1})}, \tag{4}$$

where $s_m(s_{m+1})$ denotes the length of the chord $AB\,(BC)$ and the arc length PBQ is approximated by $(s_m + s_{m+1})/2$. Regarding BQ as a circular arc of radius $|r_m|$ and length $s_{m+1}/2$, the angle γ_m may be approximated by $s_{m+1}/2|r_m|$. The direction of the normal at B is then given by

$$\theta_m = \psi_{m+1} + \gamma_m \sin r_m = \psi_{m+1} + s_{m+1}/2r_m. \tag{5}$$

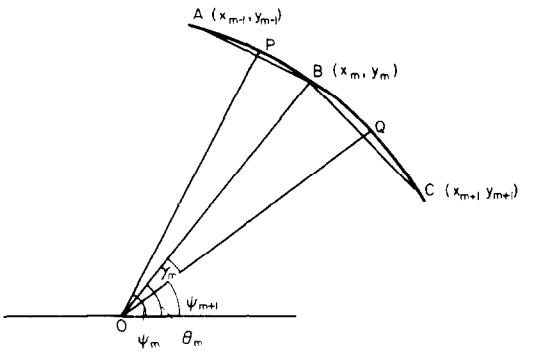

FIG. 1. Sketch illustrating the geometrical procedure used to determine the normal to each isotherm.

4. AN IMPLICIT NUMERICAL METHOD

In this section we describe an implicit scheme for the solution of a linearised form of equation (3). Consider at time $i\delta t$ three adjacent isotherms with temperatures $(j-1)\delta u$, $j\delta u$, $(j+1)\delta u$, and in Fig. 2 let $A, B(x_{j,m}, y_{j,m}), C$ denote three points on the isotherm $j\delta u$, where coordinates are known. The points F, G may be found by calculating the points of intersection of the radius at B with the chords approximating the isotherms $(j+1)\delta u$, $(j-1)\delta u$. Let $n_{j,m} = |r_{j,m}|$ be the distance of B from the centre of curvature of arc ABC, and let $n^+_{j,m}$, $n^-_{j,m}$ denote the distance of F, G respectively from the same centre of curvature. For any distance f let Δf denote the increment of f from $i\delta t$ to $(i+1)\delta t$, where δt is the size of the time step.

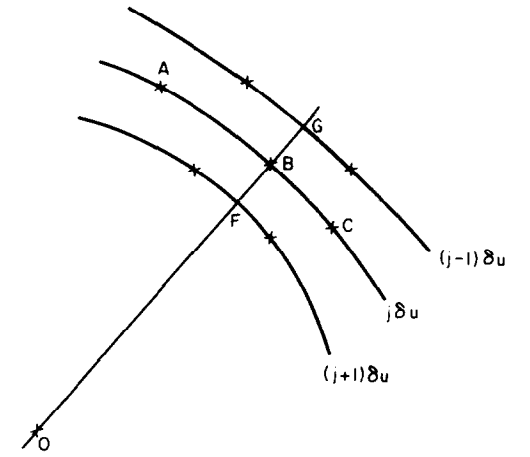

FIG. 2. Sketch showing the relative positions of isotherms.

A Crank–Nicolson scheme for equation (3) may be written as

$$\Delta n_{j,m} = \frac{\delta t}{2}\left\{\frac{\partial^2 n}{\partial u^2} \Big/ \left(\frac{\partial n}{\partial u}\right)^2 - \frac{1}{n}\right\}_{(i+1)\delta t} + \frac{\delta t}{2}\left\{\frac{\partial^2 n}{\partial u^2} \Big/ \left(\frac{\partial n}{\partial u}\right)^2 - \frac{1}{n}\right\}_{i\delta t}. \tag{6}$$

Then, at time $i\delta t$, the finite-difference approximations are

$$\left(\frac{\partial n}{\partial u}\right)_B = \frac{n^+_{j,m} - n^-_{j,m}}{2\delta u}, \tag{7}$$

and

$$\left(\frac{\partial^2 n}{\partial u^2}\right)_B = \frac{n^+_{j,m} - 2n_{j,m} + n^-_{j,m}}{(\delta u)^2}. \tag{8}$$

At $(i+1)\delta t$, each of the quantities $n_{j,m}$, $n^+_{j,m}$, $n^-_{j,m}$ has changed due to the motion of the isotherms, and we have instead

$$\left(\frac{\partial n}{\partial u}\right)_B = \frac{n^+_{j,m} - n^-_{j,m} + \Delta n^+_{j,m} - \Delta n^-_{j,m}}{2\delta u} \tag{9}$$

and

$$\left(\frac{\partial^2 n}{\partial u^2}\right)_B = \frac{n^+_{j,m} - 2n_{j,m} + n^-_{j,m} + \Delta n^+_{j,m} - 2\Delta n_{j,m} + \Delta n^-_{j,m}}{(\delta u)^2}. \tag{10}$$

Substituting from (7)–(10) into (6) and retaining only the first-order terms in the small quantities, we obtain

$$\Delta n_{j,m} = \frac{4\delta t(n^+_{j,m} - 2n_{j,m} + n^-_{j,m})}{(n^+_{j,m} - n^-_{j,m})^2}\left(1 - \frac{\Delta n^+_{j,m} - \Delta n^-_{j,m}}{(n^+_{j,m} - n^-_{j,m})}\right) + \frac{2\delta t(\Delta n^+_{j,m} - 2\Delta n_{j,m} + \Delta n^-_{j,m})}{(n^+_{j,m} - n^-_{j,m})^2} - \frac{\delta t}{n_{j,m}}\left(1 - \frac{\Delta n_{j,m}}{2n_{j,m}}\right). \tag{11}$$

If we can write $\Delta n^+_{j,m}$ in terms of $\Delta n_{j+1,k+1}$, $\Delta n_{j+1,k+1}$ for some points $(j+1,k)$, $(j+1,k+1)$ on the isotherm $(j+1)\delta u$, and similarly for $\Delta n^-_{j,m}$, we have then a set of linear equations to solve.

To this end consider the change in $n^+_{j,m}$, the distance of the point of intersection of the radius at $B(j,m)$ with the chords approximating the isotherm $(j+1)\delta u$ from the centre of curvature at B. Suppose F lies on the chord $s_{j+1,k+1}$ joining the points $(j+1,k+1)$, $(j+1,k)$, and that at time $(i+1)\delta t$ the chord has moved as shown in Fig. 3, the new intersection point being labelled by F'. We regard the centre of curvature at B as remaining fixed throughout the time step, and hence $\Delta n^+_{j,m}$ is the distance

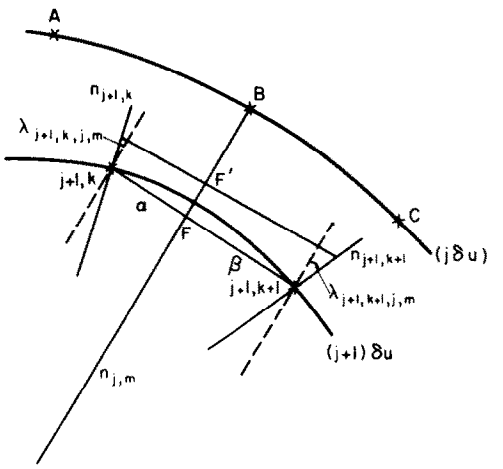

FIG. 3. Sketch illustrating the procedure used to determine $\Delta n^+_{j,m}$, $\Delta n^-_{j,m}$. The broken lines through $j+1,k$ and $j+1,k+1$ are parallel to $n_{j,m}$.

FF'. Let $\lambda_{j+1,k+1,j,m}$ denote the angle between $n_{j,m}$ and $n_{j+1,k+1}$, with $\lambda_{j+1,k,j,m}$ defined similarly, and suppose that F divide the chord $s_{j+1,k+1}$ in the ratio $\alpha:\beta$. Then

$$\Delta n^+_{j,m} = \frac{\beta}{\alpha+\beta}\Delta n_{j+1,k}\cos\lambda_{j+1,k,j,m} + \frac{\alpha}{\alpha+\beta}\Delta n_{j+1,k+1}\cos\lambda_{j+1,k+1,j,m} \tag{12}$$

and there is a similar expression involving the isotherm $(j-1)\delta u$ for $\Delta n^-_{j,m}$.

Thus, on substituting for $\Delta n^+_{j,m}$, $\Delta n^-_{j,m}$ in (11) we obtain a set of linear equations for the increments $\Delta n_{j,m}$.

If the coordinates of the point m on the isotherm $j\delta u$ at time $i\delta t$ are denoted by $x^i_{j,m}$, $y^i_{j,m}$, then the new coordinates at time $(i+1)\delta t$ are

$$x^{i+1}_{j,m} = x^i_{j,m} + \Delta n^i_{j,m}\cos\theta^i_{j,m} \tag{13}$$

$$y^{i+1}_{j,m} = y^i_{j,m} + \Delta n^i_{j,m}\sin\theta^i_{j,m}, \tag{14}$$

where $\Delta n^i_{j,m}$ is given by the solution of equation (11) and $\theta^i_{j,m}$ by equation (5).

Thus if initial data are given as the coordinates of a number of points along each of a set of isotherms, we can use the method described here to advance the isotherms in a succession of time steps δt. It should be noted that the equations derived above remain valid if any of the isotherms is, or becomes a straight line. In this case (4) and (5) imply that $\theta_m = \psi_m = \psi_{m+1}$. In (7)–(10) only the differences $n^+_{j,m} - n_{j,m}$, $n_{j,m} - n^-_{j,m}$ appear, and these are calculated directly from the coordinates of the points B, F, G. Hence the calculation procedure remains unchanged, and in this case no approximation is made in regarding the isotherm as being composed of straight line segments.

5. MOVING BOUNDARIES

Stefan problems, that is problems of heat conduction with change of phase, are of practical importance, and have attracted considerable interest in recent years (see for example [4]). In such a problem two boundary conditions are prescribed on the unknown phase-change interface, which moves through the material. These conditions are that the interface is at the melting temperature, and an expression of the heat balance across the interface. If

the interface is given by $S(\mathbf{r},t) = 0$, these conditions may be written in non-dimensional form as

$$\left.\begin{aligned} u &= 0 \\ L\frac{\partial S}{\partial t} &= \nabla u \cdot \nabla S|_- - \nabla u \cdot \nabla S|_+ \end{aligned}\right\} \text{on } S(\mathbf{r},t) = 0, \quad \begin{matrix}(15)\\(16)\end{matrix}$$

where $|_+(|_-)$ denotes the limit as S is approached from the liquid (solid) phase.

As the interface is an isotherm, these conditions take a particularly simple form in the IMM formulation. The condition (16) may be written as

$$L\frac{\partial n}{\partial t} = \left(\frac{\partial n}{\partial u}\right)^{-1}\Bigg|_- - \left(\frac{\partial n}{\partial u}\right)^{-1}\Bigg|_+, \quad \text{on } u = 0, \quad (17)$$

where n is the local radial coordinate, measuring the distance from the local centre of curvature. This expression is then used instead of equation (3) to calculate the motion of the isotherm $u = 0$.

Consider for simplicity a one-phase problem in which the solid phase is at the melting temperature $u = 0$. Using the notation of the previous section, at the point m on $u = 0$, (17) may be written as

$$\Delta n^i_{j,m} = -\frac{\delta t}{2L}\left(\left(\frac{\partial n}{\partial u}\right)^{-1}\Bigg|_{(i+1)\delta t} + \left(\frac{\partial n}{\partial u}\right)^{-1}\Bigg|_{i\delta t}\right). \quad (18)$$

Then

$$\left.\frac{\partial n}{\partial u}\right|_{i\delta t} = \frac{n^+_{j,m} - n_{j,m}}{\delta u},$$

$$\left.\frac{\partial n}{\partial u}\right|_{(i+1)\delta t} = \frac{n^+_{j,m} - n_{j,m} + \Delta n^+_{j,m} - \Delta n_{j,m}}{\delta u},$$

and substituting in equation (18) yields

$$\Delta n_{j,m} = -\frac{\delta t \cdot \delta u}{L(n^+_{j,m} - n_{j,m})}\left(1 - \frac{\Delta n^+_{j,m} - \Delta n_{j,m}}{2(n^+_{j,m} - n_{j,m})}\right), \quad (19)$$

where $\Delta n^+_{j,m}$ is calculated from equation (12) as for the other isotherms. The new coordinates of the point $(0,m)$ at time $(i+1)\delta t$ are calculated using equations (13) and (14).

6. AN EXAMPLE

The numerical method described in the previous sections was used to calculate the solution to a two-dimensional one-phase Stefan problem. The example chosen is one of those solved by Jamet and Bonnerot [5], with whose results those calculated by the IMM, using both the implicit technique of this paper and the explicit scheme discussed in [3], are compared in the next section. The rectangular region $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 4$ initially contains ice at the melting temperature in $y > 2 + \cos \pi x$, and in $y \leqslant 2 + \cos \pi x$ water with temperature distribution

$$u(x, y, 0) = 1 - \frac{y}{2 + \cos \pi x}. \quad (20)$$

All the sides are insulated, except $y = 0$ where the temperature $u = 1$ is maintained. Thus the equations are

$$\frac{\partial u}{\partial t} = \nabla^2 u, \quad (21)$$

with

$$\left.\begin{aligned} u &= 0, \\ \frac{\partial S}{\partial t} &= \nabla u \cdot \nabla S|_+ \end{aligned}\right\} \text{on } S(x, y, t) = 0, \quad (22)$$

and the boundary conditions are

$$\left.\begin{aligned} u &= 1 \text{ on } y = 0, \\ u_x &= 0 \text{ on } x = 0,\ x = 1;\ u_y = 0 \text{ on } y = 4, \end{aligned}\right\} \quad (23)$$

with the initial temperature distribution (20).

This problem is more exacting than the example to which the IMM was applied in [3] for several reasons. Firstly, the radius of curvature is positive at the left-hand end of each isotherm, and negative as the isotherm approaches $x = 1$. Therefore, proceeding in the direction of x increasing along any isotherm the curvature increases from finite positive values, through infinity, and then from negative infinity to finite negative values, the central part of each isotherm being virtually straight. Care is necessary to ensure that the finite-difference equations are written correctly when $r_{j,m} < 0$ and $n_{j,m} = -r_{j,m}$. The second difficulty arises from the nature of the initial distribution. If we concentrate attention first on the left-hand boundary $x = 0$, we can identify in (3) $\partial r/\partial t$ with $\partial y/\partial t$ since on $x = 0$ the isotherms are concave downwards and the centres of curvature are on $x = 0$. When the necessary derivatives are obtained from equation (20) and substituted into equation (3) we find that

$$\left.\frac{\partial y}{\partial t}\right|_{x=0} = -\pi^2(1 - y), \leqslant 0, 0 < u \leqslant 1.$$

However, from the heat balance condition (22) for the phase change interface $u = 0$ we have

$$\left.\frac{\partial y}{\partial t}\right|_{x=0} = +\frac{1}{3}.$$

This means that, at $t = 0$, all the isotherms except $u = 0$ initially move in the direction of y decreasing at the left-hand end, whereas the interface, $u = 0$, moves upwards. In fact, we have a discontinuity in $\partial y/\partial t$ at $t = 0$ on the boundary, $x = 0$, and we must expect our finite-difference formulae to suffer a loss of accuracy there. To be more precise, an uneven spatial distribution of the isotherms develops in this region, there being a zone of nearly zero temperature gradient behind the interface near $x = 0$. It was found that adding an extra isotherm with temperature $\frac{1}{2}\delta u$ improved the results.

A similar analysis of isotherm movement at $t = 0$ on the right-hand side, $x = 1$, shows that all isotherms move upwards initially. For $0 < u \leqslant 1$, $\partial y/\partial t|_{x=1} = \pi^2(1 - u)$ and for $u = 0$, $\partial y/\partial t|_{x=1} = 1$. These statements about isotherm movements are consistent with $\partial u/\partial t$, calculated at $t = 0$ from (20) and (21), which is negative on $x = 0$ and positive on $x = 1$ for $0 < u \leqslant 1$. Physically, we have at $t = 0$ a temperature gradient from left to right along any line of constant y and it is the associated sideways heat flow which causes the isotherms to move downwards on $x = 0$ and upwards on $x = 1$.

Table 1. Comparison of results obtained for the position of the front with 10 or 11 isotherms, implicit and explicit schemes. (The last column shows the results obtained using a modified formula involving three isotherms near the front)

		Implicit		Explicit		
	t	11 isotherms	10 isotherms	11 isotherms	10 isotherms	11 isotherms 3 pt. formula
$x=0$	0	3	3	3	3	3
	0.2	3.022	3.030	3.022	3.030	3.018
	0.4	3.034	3.048	3.034	3.048	3.027
	0.6	3.046	3.067	3.046	3.066	3.037
	0.8	3.059	3.086	3.059	3.085	3.047
	1.0	3.073	3.106	3.073	3.105	3.057
	1.2	3.088	3.127	3.087	3.126	3.069
	1.4	3.104	3.150	3.103	3.148	3.083
	1.6	3.122	3.173	3.120	3.171	3.097
	1.8	3.141	3.198	3.139	3.196	3.113
	2.0	3.161	3.224	3.159	3.221	3.131
$x=1$	0	1	1	1	1	1
	0.2	1.456	1.406	1.449	1.372	1.489
	0.4	1.707	1.642	1.683	1.606	1.720
	0.6	1.876	1.812	1.850	1.777	1.886
	0.8	2.013	1.951	1.987	1.918	2.019
	1.0	2.132	2.073	2.106	2.041	2.133
	1.2	2.238	2.183	2.211	2.153	2.239
	1.4	2.336	2.283	2.309	2.257	2.337
	1.6	2.427	2.377	2.400	2.352	2.426
	1.8	2.511	2.466	2.485	2.442	2.512
	2.0	2.591	2.550	2.567	2.526	2.594

Table 2. Comparison of positions of the interface ends obtained by various methods. (The results in the first three columns were obtained using the techniques described in [5, 6, 7] respectively, those in the fourth column by the implicit IMM introduced here and those in the fifth column by the explicit IMM [3])

On $x=0$					
t	J & B [5]	RMF [6]	Enthalpy [7]	IMM 11 isotherms implicit	IMM 11 isotherms 3 pt. formula explicit
0	3	3	3	3	3
1	3.021	3.015	3.031	3.073	3.057
2	3.068	3.054	3.051	3.161	3.131
On $x=1$					
0	1	1	1	1	1
1	2.118	2.095	2.124	2.132	2.133
2	2.610	2.585	2.566	2.591	2.594

To solve equations (21)–(23) the equations were rewritten in the IMM form, with the initial isotherm positions calculated from equation (20). Initially $C+1$ points (0–C) were taken at equal intervals in x on each of $B+1$ isotherms. The local centre and radius of curvature is calculated at each point as in Section 3, in order to set up the equations for the increments $\Delta n_{j,m}$, (11) being used within the liquid phase, and (19) on the interface $u=0$. These equations are then solved using Gauss–Seidel iteration, and the new coordinates are calculated from (13) and (14). These two parts, the geometrical procedure followed by the solution of the equations and calculation of the new isotherm positions, are repeated at each time step.

The motion of the endpoints of the isotherms needs special treatment. Since each of the sides $x=0$, $x=1$ is thermally insulated the heat flow at the endpoints of the isotherms is in the y direction. The radius of curvature at the endpoint of each isotherm is calculated by fitting a circle centred on $x=0$ (respectively $x=1$) through the endpoint and

the next interior point on the isotherm, as described in [3].

If points on a given isotherm move too close together, they must be respaced. In [3] the respacing was carried out when the distance between two neighbouring points decreased below a certain limit. However in this problem, where the ends of the isotherms remain on $x = 0$, $x = 1$ throughout, it is found that better results are obtained if the points are respaced whenever the difference in the x coordinates of successive points falls below a certain limit. This prevents the occurrence of a situation where with m increasing from $x = 0$ to $x = 1$, $x_{j,m} > x_{j,m+1}$ which is not prevented by the earlier condition if $y_{j,m}$ and $y_{j,m+1}$ are sufficiently different. The respacing algorithm is the same as in [3]; if $x_{j,m}$ moves too close to $x_{j,m+1}$, the coordinates of the point (j,m) are replaced by those of the midpoint of the chord $s_{j,m}$ *joining* (j,m) to $(j,m-1)$.

7. RESULTS AND DISCUSSION

In Table 1 a comparison is made of the position of the moving boundary on the lines $x = 0$, $x = 1$ as calculated using 10 and 11 isotherms, the extra one being that with temperature $\delta u/2$, and 11 points on each isotherm. The results obtained using both the implicit scheme with $\delta t = 0.0008$ and 10 isotherms, and the explicit scheme with $\delta t = 0.0004$ and 10 or 11 isotherms, are given. In the last column of Table 1 the results obtained using a three isotherm formula for $\partial n/\partial u$ at the phase change front are given. The computation time on a CDC 7600 was about 320 s for the explicit scheme with 11 isotherms, 290 s with 10 isotherms. For the implicit scheme with 10 isotherms the time was 220 s, but with 11 isotherms was 810 s. The insertion of the extra isotherm necessitates a much smaller time step in order that the iterative scheme used for solving equations (11) and (19) will converge near the front on $x = 1$. The time step used was $\delta t = 0.0001$ until $t = 0.1$, $\delta t = 0.00025$ thereafter. The number of iterations was typically only three. Thus, with the extra isotherm, the usual advantage of an implicit scheme—a larger time step—is lost, and the computation time is greatly increased.

In Table 2 some results obtained for this problem by other methods are shown for comparison. It is immediately seen that, for small times, the boundary as calculated by IMM moves rather fast on $x = 0$, and rather slowly on $x = 1$. This is caused by using points too far from the moving boundary in evaluating the heat flux there, and results in an over-estimate of the flux on $x = 0$, where the temperature profile is concave upwards, and an under-estimate on $x = 1$ where the temperature profile is convex upwards. It is this observation that indicates the need for the extra isotherm with temperature $\delta u/2$, and explains the improvement shown in the results in Table 1.

We have explained in Section 6 above the exacting nature of the example chosen which was not previously suspected. The presence of the discontinuity in isotherm velocity at $t = 0$, particularly on the boundary $x = 0$, makes this region an undesirable one in which to compare methods and obscures the relative merits of the implicit and explicit algorithms. However, experience in other parts of the domain indicates that in general the implicit scheme offers the usual advantage over an explicit one in enabling the use of longer time steps, especially if a more efficient iterative scheme were adopted.

Acknowledgement—One of us (A.B.C.) wishes to thank the Science Research Council and the Royal Commission for the Exhibition of 1851 for financial support.

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SUR UN SCHEMA IMPLICITE POUR LA METHODE DE MIGRATION ISOTHERME LE LONG DES LIGNES ORTHOGONALES DANS DEUX DIMENSIONS

Résumé—Dans un article précédent les auteurs ont développé un schéma explicite pour la solution numérique des équations de la migration isotherme le long des lignes orthogonales de déplacement dans deux dimensions spatiales. On présente ici un schéma implicite. On l'utilise pour résoudre un problème bidimensionnel de Stefan avec une phase et les résultats sont comparés avec ceux obtenus par d'autres méthodes.

ZU EINEM IMPLIZITEN VERFAHREN FÜR DIE ISOTHERMEN-MIGRATIONS-METHODE LÄNGS ORTHOGONALER STROMLINIEN IN ZWEI DIMENSIONEN

Zusammenfassung—In einer früheren Veröffentlichung entwickelten die Autoren ein explizites Verfahren zur numerischen Lösung der Gleichungen der Isothermen-Ausbreitung längs orthogonaler Stromlinien in zwei räumlichen Dimensionen. In der vorliegenden Arbeit wird ein implizites Verfahren vorgestellt. Es wird zur Lösung eines zweidimensionalen Einphasen-Stefan-Problems benutzt; die Ergebnisse werden mit denjenigen nach anderen Methoden verglichen.

О НЕЯВНОЙ СХЕМЕ ЧИСЛЕННОГО МЕТОДА РЕШЕНИЯ ДВУМЕРНОЙ ЗАДАЧИ О МИГРАЦИИ ИЗОТЕРМ ВДОЛЬ ОРТОГОНАЛЬНЫХ ЛИНИЙ ТОКА

Аннотация — В предыдущей работе авторы разработали явную схему численного решения двумерных уравнений миграции изотерм вдоль ортогональных линий тока. В данной статье представлена неявная схема, которая используется при решении двумерной однофазной задачи Стефана. Приведено сравнение с результатами, полученными при использовании других методов.